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A note on hyperdegrees

by

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1. Introduction. Following [2], for a given class $S\subseteq\omega^{\omega}$, let $U(S)=\{X\subseteq\omega\mid\exists\alpha\in S[\alpha\leq_{T}X]\}$ and $D(S)=\{X\subseteq\omega\mid\exists\alpha\in S[\alpha=_{T}X]\}$, where $\alpha\leq_{T}\beta$ means α is Turing reducible to β and $\alpha=_{T}\beta$ denotes $\alpha\leq_{T}\beta\wedge\beta\leq_{T}\alpha$. Jockusch and Soare have shown the following theorems among other things [2]: (i) *If S is a Π_{1}^{0} class which has no recursive member, then $U(S)$ (and hence $D(S)$) is meager.* (ii) *If A and B are disjoint recursively inseparable sets of natural numbers and if $S=\{C_{X}\mid X$ separates A and $B\}$, where C_{X} denotes the characteristic function of a set $X\subseteq\omega$, then $U(S)$ has measure* 0. (iii) *There exists a simple set $A\subseteq\omega$ such that $D(S)$ has measure* 1, *where $S=\{C_{X}\mid X\subseteq\omega-A\}$.*

Now for a class $S\subseteq\omega^{\omega}$, let $U^{*}(S)=\{X\mid\exists\alpha\in S[\alpha\leq_{H}X]\}$ and $D^{*}(S)=\{X\mid(\exists\alpha\in S)[\alpha=_{H}X]\}$, where $\alpha\leq_{H}\beta$ means α is hyperarithmetic in β and $\alpha=_{H}\beta$ denotes $\alpha\leq_{H}\beta\wedge\beta\leq_{H}\alpha$. In this note we deduce the hyperarithmetic counterparts of (i)-(iii) which are obtained by replacing 'recursive' by 'hyperarithmetic'. We write HA for 'hyperarithmetic' and $\mu(S)$ for the measure of a Lebesgue measurable S.

2. *Let S be a Π_{1}^{1} class which has no HA member. Then $U^{*}(S)$ (and hence $D^{*}(S)$) is meager* [Counterpart of (i)].

This is a corollary of Hinman's theorem [1] asserting that every non-meager Π_{1}^{1} class has a HA element, since $U^{*}(S)$ is Π_{1}^{1} and it has no HA member.

Also it is known by Jockusch that $U^{*}(S)$ has measure 0. [6; p. 394].

3. THEOREM. *Let A and B be disjoint, HA-inseparable sets of natural numbers and let*

$$S=\{C_{Y}\mid Y\ (\subseteq\omega)\ \text{separates}\ A\ \text{and}\ B\}\ ,$$

where C_{Y} is the characteristic function of Y. Then $U^{*}(S)$ has measure 0 [Counterpart of (ii)].

Proof. Let $S(a,b)=\{X\subseteq\omega\mid a\in O^{X}\wedge[\{b\}^{H_{a}^{X}}$ is the characteristic function of a set$]\wedge\{b\}^{H_{a}^{X}}\in S\}$. Then $U^{*}(S)=\bigcup_{a=0}^{\infty}\bigcup_{b=0}^{\infty}S(a,b)$. If we suppose $\mu(U^{*}(S))>0$, we can find a and b such that $\mu(S(a,b))>0$. For the sake of simplicity we set $S'=S(a, b)$. Then there is a finite $\{0, 1\}$-sequence σ such that

$$\mu([\sigma]\cap S')>\tfrac{2}{3}\times 2^{-lh(\sigma)}\ ,$$

where $[\sigma]=\{X\subseteq\omega\mid \bar{C}_X(lh(\sigma))=\sigma\}$. Here we regard σ as a sequence number. Let

$$S_n^{(i)}=\{X\subseteq\omega\mid a\in O^X\wedge\{b\}^{H_a^X}(n)=i\}$$

for $i=0, 1$, and let

$$D_i=\{n\mid\mu([\sigma]\cap S_n^{(i)})>\tfrac{1}{3}\times 2^{-lh(\sigma)}\}$$

for $i=0, 1$. By [7], if $P(n, X)$ is Π_1^1, then $\lambda nr[\mu(\{X\mid P(n, X)\})>r]$ is also Π_1^1, where r ranges over the rational numbers. Therefore both D_0 and D_1 are Π_1^1. On the other hand, since $S_n^{(0)}\cup S_n^{(1)}$ contains S', $\mu([\sigma]\cap(S_n^{(0)}\cup S_n^{(1)}))>\tfrac{2}{3}\times 2^{-lh(\sigma)}$. So $D_0\cup D_1=\omega$. By applying the reduction theorem for Π_1^1 sets, we can find disjoint Π_1^1 sets E_0 and E_1 such that $E_i\subseteq D_i$ and $E_0\cup E_1=D_0\cup D_1=\omega$. So E_0 and E_1 are HA. Furthermore, it can be proved that $A\subseteq E_1$ and $B\subseteq E_0$. This contradicts the hypothesis that A and B are HA-inseparable.

COROLLARY (Sacks [6]). *If A is a non HA set of natural numbers, then* $\mu(\{X\subseteq\omega\mid A\leq_H X\})=0$.

4. A set A of natural numbers is called *HA-simple* if A is Π_1^1, its complement $\bar{A}$ is infinite and $\bar{A}$ contains no infinite Π_1^1 subsets [5; p. 449].

THEOREM. *There exists a HA-simple set A such that* $\mu(D^*(S))=1$, *where* $S=\{C_X\mid X\subseteq\bar{A}\}$ [Counterpart of (iii)].

Proof. Let $\Sigma^*=\{\sigma_0, \sigma_1, \cdots, \sigma_n, \cdots\}$ be an effective enumeration of all finite $\{0, 1\}$-sequences (without repetition). Let P_e be the e-th Π_1^1 subset of ω. We may assume $\lambda ex[x\in P_e]$ is also Π_1^1. Let S $(\subseteq\omega\times\omega\times\Sigma)$ be a recursive sieve for P, where Σ is the set of all finite sequences of natural numbers. Then

$$\begin{aligned}x\in P_e\Leftrightarrow[&S^{\langle e,x\rangle}=\{\sigma\in\Sigma\mid\langle e, x, \sigma\rangle\in S\}\text{ is well-ordered}\\&\text{by the Kleene-Brouwer linear ordering }\prec].\end{aligned}$$

We shall define $Q(e, x)$ as follows (a slight modification of the definition used in Kreisel [3]):

$$\begin{aligned}Q(e, x)\Leftrightarrow\ & x\in P_e\wedge lh(\sigma_x)>e\\&\wedge(\forall y)[lh(\sigma_y)>e\rightarrow|S^{\langle e,y\rangle}|\not<|S^{\langle e,x\rangle}|]\\&\wedge(\forall y)[lh(\sigma_y)>e\wedge y<x\rightarrow|S^{\langle e,y\rangle}|\not\leq|S^{\langle e,x\rangle}|]\end{aligned}$$

where $|*|$ denotes the order-type of the set $*$ with respect to $\prec$.

Define $A=\bigcup_{e=0}^{\infty}\{x \mid Q(e,x)\}$. Then A is Π_1^1. $\bar{A}$ is infinite, since $A^*=\{\sigma_x \mid x\in A\}$ has at most e finite sequences of length e for each e. Furthermore $\bar{A}$ contains no infinite Π_1^1 subsets. For, let B be an arbitrary infinite Π_1^1 subset of ω. Take an e such that $B=P_e$. Since P_e is infinite, there is an $x\in P_e$ such that $lh(\sigma_x)>e$. (Note that each σ_x is a $\{0,1\}$-sequence.) So

$$x_0=(\mu x)[lh(\sigma_x)>e\wedge x\in P_e \wedge(\forall y)(lh(\sigma_y)>e\wedge y\in P_e\rightarrow|S^{\langle e,x\rangle}|\leq|S^{\langle e,y\rangle}|)]$$

is well-defined. Then $Q(e,x_0)$ holds and hence $x_0\in A$. That is $B\not\subseteq\bar{A}$. Therefore A is a HA-simple set.[1] As in [2], we can show that $\mu(D(S))=1$ for $S=\{Y \mid Y\subseteq\bar{A}\}$, *a fortiori* $\mu(D^*(S))=1$.

References

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1) It is clear that every Π_1^1-maximal set in the sense of Kreisel-Sacks [4] is HA simple. Jockusch pointed out however that *this* HA simple set A is not Π_1^1-maximal.